Weights in Serre's conjecture in Hilbert modular forms

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$\rho: G_{\mathbb{Q}} \longrightarrow GL_2(\overline{\mathbb{F}}_p)$. $\qquad$ F totally real fld.

$\rho: \mathrm{Gal}(\overline{F}/F) \longrightarrow GL_2(\overline{\mathbb{F}}_p)$

Conj (?): Let ρ be continuous, irred., totally odd, then ρ is modular.

(Serre) wts are irred $\overline{\mathbb{F}}_p$-rep'ns of $GL_2(\mathcal{O}_F/p) = \prod_{\mathfrak{p}|p} GL_2(\mathcal{O}_F/\mathfrak{p}_i^{e_i})$

$p\mathcal{O}_F = \mathfrak{p}_1^{e_1} \cdots \mathfrak{p}_r^{e_r}$

Wts factor through the quotient $\prod_{\mathfrak{p}_i|p} GL_2(k_{\mathfrak{p}_i})$

Wts: $GL_2(\mathbb{F}_p)$ $\qquad$ $\det^w \otimes \mathrm{Sym}^{k-2}\, \overline{\mathbb{F}}_p^2$ $\qquad$ $0 \le w \le p-2$ $\qquad$ $2 \le k \le p+1$

$GL_2(\mathbb{F}_q)$ $\qquad$ $\bigotimes_{\tau: \mathbb{F}_q \hookrightarrow \overline{\mathbb{F}}_p} \det^{w_\tau} \otimes \left(\mathrm{Sym}^{k_\tau - 2}\, \mathbb{F}_q^2 \otimes_\tau \overline{\mathbb{F}}_p\right)$

B/F quaternion alg, split everywhere over p and at one ∞ prime.

$G = \mathrm{Res}_{F/\mathbb{Q}} B^\times$.

$U \subseteq G(\mathbb{A}^\infty) \sim X_U \quad = \quad G(\mathbb{Q}) \backslash G(\mathbb{A}^\infty) / U$

If $H \subseteq G(\mathbb{A}^{\infty,p})$ is suff. small, then

$$X_{1,H} \downarrow X_{0,H}$$

is a Galois cover with grp $GL_2(\mathcal{O}_F/p)$

$X_{0,H} = \prod_{\mathfrak{p}|p} GL_2(\mathcal{O}_{\mathfrak{p}}) \times H$

$X_{1,H} = \prod_{\mathfrak{p}|p} \begin{pmatrix} \equiv 1 & \bmod \\ & \mathfrak{p} \end{pmatrix} \times H$.

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Def: $\rho: \mathrm{Gal}(\overline{F}/F) \to GL_2(\overline{\mathbb{F}}_p)$ is **modular** of wt σ if

$$\rho \subseteq \underbrace{\left(\mathrm{Pic}^0(X_{I,H})_{\overline{\mathbb{F}}_p}[p^\infty] \otimes \sigma\right)}_{p\text{-torsion part} \ldots}{}^{GL_2(\mathcal{O}_F/p)}.$$

Thm (Fontaine, '79) Let $\rho: G_{\mathbb{Q}} \to GL_2(\overline{\mathbb{F}}_p)$ be modular of wt. $\det^w \otimes \mathrm{Sym}^{k-2}\overline{\mathbb{F}}_p^2$

$\rho|_{I_p}$ is irred. Then

$$\rho|_{I_p} \sim \omega^w \otimes \begin{pmatrix} \omega_2^{k-1} & \\ & (\omega_2^p)^{k-1} \end{pmatrix}$$

$W(\rho)$ modular wts of ρ

For each $\wp | p$, we'll define a set $W_\wp^?(\rho)$ of rep'ns of $GL_2(k_\wp)$

Conj that $W(\rho) = \{ \sigma = \bigotimes_{\wp | p} \sigma_\wp \mid \sigma_\wp \in W_\wp^?(\rho) \}$

Assume: there is only one place of F dividing p. $p\mathcal{O}_F = \wp^e$.

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Conj: Suppose $k_{\wp} = \mathbb{F}_p$.

Given $\rho : Gal(\overline{F}/F) \longrightarrow Gl_2(\overline{\mathbb{F}}_p)$

that is locally irred at $\wp | p$. then

ρ is modular of wt $\det^w Sym^{k-2} \overline{\mathbb{F}}_p^2 \iff \exists\ 0 \le \delta \le e-1$

s.t.

$$\rho|_{I_p} \sim \omega^w \begin{pmatrix} \omega_2^{k-1+\delta} (\omega_2^p)^{e-1-\delta} & \\ & (\quad '' \quad)^p \end{pmatrix}$$

With ext'n of residue flds:

Suppose $|k_{\wp}| = p^s$. For each $k_{\wp} \overset{\tau}{\hookrightarrow} \overline{\mathbb{F}}_p$, let ω_τ be the niveau s character corresponding to τ

$$I_{\wp} / I_{\wp}' = \varprojlim \mathbb{F}_{\wp^n}^\times \longrightarrow k_{\wp}^\times \xrightarrow{\tau} \overline{\mathbb{F}}_p^\times$$

Let ω_τ' be one of the two niveau $2s$ characters lifting ω_τ

Then there should exist $0 \le \delta_\tau \le e-1$, s.t.

$$\rho|_{I_{\wp}} \sim \begin{pmatrix} \prod_\tau (\omega_\tau')^{w_\tau + k_\tau - 2 + \delta_\tau} (\omega_\tau'^{p^s})^{w_\tau + e - 1 - \delta_\tau} & 0 \\ 0 & (\quad)^{p^s} \end{pmatrix}$$

Evidence: 1) Computations of L. Dembélé $F = \mathbb{Q}(\sqrt{5})$ $p=5$

2) Thm: Suppose $\rho : Gal(\overline{F}/F) \to Gl_2(\overline{\mathbb{F}}_p)$ is modular of wt $\bigotimes_\tau \det^{w_\tau} (Sym^{k_\tau - 2} k_{\wp}^2 \otimes_\tau \overline{\mathbb{F}}_p)$ and $\rho|_{I_{\wp}}$ is irred. If

$|k_{\wp}| > p$ and also suppose $k_\tau - 2 + e \le p-1$ for all τ

then $\sigma \in W^?(\rho)$.

pf: If ρ is modular of wt σ, $\exists f \in H^2_{et}(X_{0,H} \otimes \overline{F}, \mathcal{F}_\sigma)$.

where $\mathcal{F}_\sigma$ is an étale sheaf given by

⌈ where $\mathcal{F}_\sigma$ is an étale sheaf given by

$Y \to X_{0,H}$ étale cover

$$\mathcal{F}_\sigma(Y) = \{ g : Y \times_{X_{0,H}} X_{1,H} \to V_\sigma \}$$

locally const. s.t.

$\forall \; C \in \pi_0(Y \times X_{1,H}),\ \gamma \in GL_2(\mathcal{O}_F/\wp)$

$$g(C_\gamma) = \sigma(\gamma)^{-1} \cdot g(C)$$

⌊

$\exists\, f \in H^2_{\text{ét}}(X_{0,H} \otimes \overline{F}, \mathcal{F}_\sigma) \rightsquigarrow \rho_{f,p} \simeq \rho.$

Lift $\quad X^{\text{bal}}_{\mathcal{U}(\wp),H} \leftrightsquigarrow \{ M \in GL_2(\mathcal{O}_\wp) \mid M \equiv \begin{pmatrix} 1 & * \\ & 1 \end{pmatrix} \bmod \wp \} \times H$

f to $\tilde{f} \in H^1(X^{\text{bal}}_{\mathcal{U}(\wp),H}, \mathcal{F}_\sigma)$

$B(k_\wp) \subseteq GL_2(k_\wp)$ upper triangular

Choose a character $\theta : B(k_\wp) \to \overline{\mathbb{F}}_p^\times$ s.t. $\sigma \subseteq \mathrm{Ind}_{B(k_\wp)}^{GL_2(k_\wp)} \theta$.

Define a finite flat gp scheme $\mathcal{H}/_{D'}$ by picking out a piece of $\mathrm{Jac}(X^{\text{bal}}_{\mathcal{U},(\wp),H})[p^\infty]$

$K = F_\wp^{\text{nr}} \qquad \mathrm{Gal}(K'/K) \simeq k_\wp^\times$

D, D' valuation rings of K, K'.

We have two actions:

$I_\wp = \mathrm{Gal}(\overline{K}/K)$ acts on $\mathcal{H}(\overline{K})$ via φ

$\rho|_{I_\wp} = \begin{pmatrix} \varphi & \\ & \varphi' \end{pmatrix} \qquad \varphi = \omega_{2s}^{a_2 + pa_1 + p^2 a_2 + \cdots}$

$\mathrm{Gal}(K'/K)$ acts on $\mathrm{cot.}(\mathcal{H} \times_D \overline{\mathbb{F}}_p)$

$\rightsquigarrow$ parameters b_i

$k_{\mathfrak{p}} = \mathbb{F}_p$

$$a_i' = b_{i+1} - pb_i + (p-1)a_i \qquad 0 \le a_i' \le e(p-1)$$

$$\theta : \begin{pmatrix} a & b \\ & d \end{pmatrix} \mapsto a^w d^{k-2+w} \rightsquigarrow \{b_0, b_1\} = \{\overset{b_0}{\overset{\shortparallel}{w}}, \overset{b_1}{\overset{\shortparallel}{w+k-2}}\}$$

$$a_0' = k-2 + (p-1)(a_0 - w) \qquad a_0 = w, \dots, w+e-1.$$

If want to use finite flat gp schemes, then (by H-T number consideration) we have to lift to wt 2, and cannot overcome difficulties. Too many constituents in the induction...

Instead, use Toby Gee's ideas ... (Need to step out of alg. geom. ...)

Comparing with Florian's notion of modularity.

Let $\rho: G_{\mathbb{Q}} \to GL_n(\overline{\mathbb{F}}_p)$ be modular of wt $\sigma = F(a_1, \dots, a_n)$

i.e. there exist Shimura varieties (as in Harris-Taylor)

$$\begin{array}{c} X_{1,H} \\ \big\downarrow {\scriptstyle GL_2(\mathbb{F}_p)} \\ X_{0,H} \end{array} \qquad \rho|_{Gal(\bar{E}/E)} \subseteq H^{\bullet}_{ét}(X_{0,H} \otimes_E \bar{E}, \mathcal{F}_\sigma) \circlearrowleft Gal(\bar{\mathbb{Q}}/E) \qquad E_v = \mathbb{Q}_p.$$

$X_{0,H}$, $X_{1,H}$ defined over E,

$E/\mathbb{Q}$ imag. quad. ext'n.

__Def.__: $\sigma = F(a_1, \dots, a_n)$ is __p-miniscule__ if $(a_1 - a_n) + (a_2 - a_n) + \dots + (a_{n-1} - a_n) < p - n$

__Thm.__: Let $\rho: G_{\mathbb{Q}} \to GL_n(\overline{\mathbb{F}}_p)$ be modular of a p-miniscule wt.

$F(a_1, \dots, a_n)$. Then the Fontaine-Laffaille numbers of ρ are contained in the set $\{a_i + (n-i) \mid 1 \leq i \leq n\}$.

uses

Faltings's comparison thm, Tilouine's BGG cx.